Statistical properties of random density matrices

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Abstract
Statistical properties of ensembles of random density matrices are investigated.
We compute traces and von Neumann entropies averaged over ensembles of
random density matrices distributed according to the Bures measure. The
eigenvalues of the random density matrices are analysed: we derive the
eigenvalue distribution for the Bures ensemble which is shown to be broader
then the quarter-circle distribution characteristic of the Hilbert–Schmidt
ensemble. For measures induced by partial tracing over the environment we
compute exactly the two-point eigenvalue correlation function.

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1. Introduction

Analysing the density matrices of a finite size N, one is often interested in the properties of
typical states. In general the properties depend on the measure, according to which the random
density matrices are distributed. Nowadays it is widely accepted that it is not possible to single
out the only unique measure in the set $\mathcal{M}_N$ of all density matrices of size N.

However, several possible measures are distinguished by different mathematical and
physical arguments. For instance, the Hilbert–Schmidt (HS) measure arises if one constructs
random pure states $|\Psi\rangle$ distributed according to the natural (Fubini–Study) measure on the
space of pure states on a composite Hilbert space $\mathcal{H}_N \otimes \mathcal{H}_N$ and obtains mixed states by partial
tracing, $\rho = \mathrm{Tr}_N(|\Psi\rangle\langle\Psi|)$ [1]. Moreover, the HS measure may be defined by the HS distance,

$$D_{\mathrm{HS}}(\rho, \sigma) = \sqrt{\mathrm{Tr}\left[(\rho - \sigma)^2\right]}, \qquad (1.1)$$

which induces the flat geometry in $\mathcal{M}_N$. For instance, the set of $N = 2$ mixed states analysed with respect to the HS distance displays the geometry of the 3-ball (the Bloch ball), with the Bloch sphere, containing the pure states, at its boundary.

Another measure in the space of mixed states, which should be distinguished, is the Bures measure [2]. It is induced by the Bures metric [3, 4]

$$D_B(\rho, \sigma) = \sqrt{2}[1 - \text{Tr}\,(\sqrt{\rho}\sigma\sqrt{\rho})^{1/2}]^{1/2} \tag{1.2}$$

which is Riemannian and monotone. It is a Fisher-adjusted metric [5], since in the subspace of diagonal matrices it induces the statistical distance [6]. Moreover, the Bures metric is Fubini–Study adjusted, since at the space of pure states both metrics do agree [7]. These unique features of the Bures distance are used to support the claim that without any prior knowledge on a certain density matrix, the optimal way to mimic it is to generate it at random with respect to the Bures measure.

In this work we analyse statistical properties of ensembles of random states. In section 2 we provide the definitions of the Hilbert–Schmidt and the Bures ensembles of random density matrices and recall their joint distribution functions for their spectra. The expectation values of the moments $\langle \text{Tr}\,\rho^q \rangle$ and von Neumann entropies are computed in section 3. In section 4 we analyse the eigenvalue density of random states: the quarter-circle distribution characteristic of the Hilbert–Schmidt ensemble is rederived and compared with an explicit distribution computed for the Bures ensemble. Section 5 is devoted to the ensembles of random matrices obtained by partial tracing, for which the average traces and the eigenvalue correlation functions are computed.

2. Ensembles of random states

We are concerned with ensembles of random states, for which the probability measure has a product form and may be factorized [2, 1],

$$d\mu_x = d\nu_x(\lambda_1, \lambda_2, \ldots, \lambda_N) \times dh. \tag{2.1}$$

The latter factor, dh, determining the distribution of the eigenvectors of the density matrix, is the unique, unitarily invariant, Haar measure on $U(N)$. On the other hand, the first factor describing the distribution of eigenvalues λ_i of ρ depends on the measure used (the label x denotes any of the product measures investigated).

The Hilbert–Schmidt measure induces the following joint distribution function in the simplex of eigenvalues [2, 1]:

$$P_{\text{HS}}(\vec{\lambda}) = \frac{\Gamma(N^2)}{\prod_{j=0}^{N-1} \Gamma(N-j)\Gamma(N-j+1)} \delta\left(1 - \sum_{j=1}^{N} \lambda_j\right) \prod_{i<j}^{N} (\lambda_i - \lambda_j)^2. \tag{2.2}$$

This distribution may be considered as a special case of the family of measures induced by partial tracing. Consider a pure state of $|\Phi\rangle \in \mathcal{H}_N \otimes \mathcal{H}_K$ of a composite bi-partite system of size $N \times K$. Tracing over the K-dimensional environment, one obtains a mixed state of size N, namely $\rho = T_K(|\Phi\rangle\langle\Phi|)$. A natural assumption that $|\Phi\rangle$ is a random pure state distributed according to the unique, unitarily invariant measure on the set of pure states leads to the family of measures

$$P_{N,K}(\vec{\lambda}) = \frac{\Gamma(KN)}{\prod_{j=0}^{N-1} \Gamma(K-j)\Gamma(N-j+1)} \delta\left(1 - \sum_{j=1}^{N} \lambda_j\right) \prod_i \lambda_i^{K-N} \prod_{i<j} (\lambda_i - \lambda_j)^2, \tag{2.3}$$

labelled by the size K of the environment. Such induced measures have been discussed many times in the literature [8–10], while the normalization constant was derived in [1]. Note that in the symmetric case $K = N$ the induced measure reduces to the Hilbert–Schmidt measure (2.2).

It is worth emphasizing a link to known ensembles of random matrices. In order to construct a random density matrix according to the measure $\mu_{N,K}$ it is sufficient to generate a rectangular Gaussian matrix X of size $N \times K$ and to compute $\rho = X^\dagger X / \mathrm{Tr}\,(X^\dagger X)$ [1]. In the special case $K = N$ this fact shows a relation between the Ginibre ensemble of non-Hermitian random matrices [11] and the Hilbert–Schmidt measure.

The Bures measure in the simplex of eigenvalues may be derived from an assumption that any ball in the sense of the Bures distance of a fixed radius belonging to the set $\mathcal{M}_N$ has the same volume. The Bures probability distribution in the simplex of eigenvalues was obtained by Hall [2]

$$P_{\mathrm{B}}(\vec{\lambda}) = C_N \frac{\delta(\lambda_1 + \lambda_2 + \cdots + \lambda_N - 1)}{\sqrt{\lambda_1 \lambda_2, \ldots, \lambda_N}} \prod_{i<j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_j + \lambda_j}. \tag{2.4}$$

The normalization constants C_N were found by Slater [12] for low values of N, while the general formula

$$C_N = 2^{N^2 - N} \frac{\Gamma(N^2/2)}{\pi^{N/2} \prod_{j=1}^{N} \Gamma(j+1)} \tag{2.5}$$

was derived in [13]. The volume of the set of mixed quantum states and the area of its boundary were computed with respect to both measures in [14, 13].

3. Expectation values

To characterize the extent to which a given state ρ is mixed, one may use the moments, $\mathrm{Tr}\,\rho^q$, with any $q > 0$. The simplest to compute is the second trace $r = \mathrm{Tr}\,\rho^2$, called *purity*, which is closely related to the *linear entropy* $1 - r$ and inverse participation ratio, $R = 1/r$. Mean purity averaged over the HS measure is smaller then the average over the Bures measure,

$$\langle \mathrm{Tr}\,\rho^2 \rangle_{\mathrm{HS}} = \frac{2N}{N^2 + 1} < \langle \mathrm{Tr}\,\rho^2 \rangle_{\mathrm{B}} = \frac{5N^2 + 1}{2N(N^2 + 2)}. \tag{3.1}$$

This result reflects the fact that the Bures measure is more concentrated on the states of higher purity, than the Hilbert–Schmidt measure [1]. It is not so simple to get such results for an arbitrary exponent q. However, it is easier to perform averaging in the asymptotic regime, $N \gg 1$. Mean traces, averaged over the Hilbert–Schmidt measure are

$$\langle \mathrm{Tr}\,\rho^q \rangle_{\mathrm{HS}} = N^{1-q} \frac{\Gamma(1 + 2q)}{\Gamma(1 + q)\Gamma(2 + q)} \left(1 + O\left(\frac{1}{N}\right)\right). \tag{3.2}$$

The analogous average with respect to the Bures measure reads

$$\langle \mathrm{Tr}\,\rho^q \rangle_{\mathrm{B}} = N^{1-q} 2^q \frac{\Gamma[(3q+1)/2]}{\Gamma[(1+q)/2]\Gamma(2+q)} \left(1 + O\left(\frac{1}{N}\right)\right). \tag{3.3}$$

Again we find $\langle \mathrm{Tr}\,\rho^q \rangle_{\mathrm{HS}} < \langle \mathrm{Tr}\,\rho^q \rangle_{\mathrm{B}}$. As a measure of the degree of mixing one often uses the von Neumann entropy, $S(\rho) = -\mathrm{Tr}\,\rho \ln \rho$. It varies from $S(\rho) = 0$ for any pure state and $S(\rho) = \ln N$ for the maximally mixed state. Since $S(\rho) = -\lim_{q \to 1} \partial\,\mathrm{Tr}\,\rho^q / \partial q$ the mean von Neumann entropy may be obtained by differentiation of (3.2) and (3.3) with respect to the

parameter, $\langle S \rangle = -\lim_{q \to 1} \langle \partial \operatorname{Tr} \rho^q / \partial q \rangle = -\lim_{q \to 1} \partial \langle \operatorname{Tr} \rho^q \rangle / \partial q$. The results are

$$\langle S(\rho) \rangle_{\mathrm{HS}} = \ln N - \frac{1}{2} + O\left(\frac{\ln N}{N}\right) \tag{3.4}$$

for the Hilbert–Schmidt measure and

$$\langle S(\rho) \rangle_{\mathrm{B}} = \ln N - \ln 2 + O\left(\frac{\ln N}{N}\right) \tag{3.5}$$

for the Bures measure. Note that the former result is larger, since the HS measure favours more mixed states. Although the mean value of the traces with respect to the HS measure have appeared several times in the literature [8–10], the results for the Bures measure are new. Their derivation is sketched in the appendix. Using the expansion of the generating functions there, it is possible to give some more moments for the Bures measure. We compare them with the previously known Hilbert–Schmidt averages:

$$\langle \operatorname{Tr} \rho^3 \rangle_{\mathrm{HS}} = \frac{5N^2 + 1}{(N^2 + 1)(N^2 + 2)}, \qquad \langle \operatorname{Tr} \rho^3 \rangle_{\mathrm{B}} = \frac{8N^2 + 7}{(N^2 + 2)(N^2 + 4)}, \tag{3.6}$$

$$\langle \operatorname{Tr} \rho^4 \rangle_{\mathrm{HS}} = \frac{14N^3 + 10N}{(N^2 + 1)(N^2 + 2)(N^2 + 3)}, \qquad \langle \operatorname{Tr} \rho^4 \rangle_{\mathrm{B}} = \frac{21(11N^4 + 25N^2 + 4)}{8N(N^2 + 2)(N^2 + 4)(N^2 + 6)}. \tag{3.7}$$

4. Distribution of eigenvalues

We are going to evaluate the distribution of the rescaled eigenvalue $x := N\lambda_1$ in the limit of large dimension N of density matrices. To derive the probability distribution $P(x)$, we analyse qth moments of these distributions. For the Hilbert–Schmidt measure we obtain

$$f_{\mathrm{HS}}(q) = \int P_{\mathrm{HS}}(x) x^q \, \mathrm{d}x = \frac{1}{\pi} 2^{2q} \frac{\Gamma(q + 1/2)\Gamma(1/2)}{\Gamma(q + 2)}, \tag{4.1}$$

while the moments for the Bures measure are

$$f_{\mathrm{B}}(q) = \int P_{\mathrm{B}}(x) x^q \, \mathrm{d}x = \frac{1}{2\pi} 2^q 3^{3q/2} \frac{\Gamma(q/2 + 1/6)\Gamma(q/2 + 5/6)}{\Gamma(q + 2)}. \tag{4.2}$$

The above results follow from equations (3.2), (3.3) by use of the duplication and triplication formula for the Gamma function. They allow us to obtain the explicit form of the level density, exact in the asymptotic limit of large N. The distribution obtained for the HS measure

$$P_{\mathrm{HS}}(x) = \frac{1}{2\pi} \sqrt{\frac{4}{x} - 1} \qquad \text{for} \quad x \in [0, 4] \tag{4.3}$$

diverges as $x^{-1/2}$ for $x \to 0$ and becomes a quarter-circle law in the rescaled variable, $y = \sqrt{x}$; see figure 1. It is comforting to verify that this law forms a special case of the distribution obtained by Page for the induced measures [10] and later derived in a different context in [15]. On the other hand, the distribution for the Bures measure

$$P_{\mathrm{B}}(x) = \frac{3}{4a\pi} \left[\left(\frac{a}{x} + \sqrt{\left(\frac{a}{x}\right)^2 - 1} \right)^{2/3} - \left(\frac{a}{x} - \sqrt{\left(\frac{a}{x}\right)^2 - 1} \right)^{2/3} \right] \qquad \text{for} \quad x \in (0, a] \tag{4.4}$$

is defined on a larger support, $x \leqslant a = 3\sqrt{3}$, and diverges for $x \to 0$ as $x^{-2/3}$. Level repulsion for the Bures ensemble compared to the HS ensemble will be reduced at $x = 0$ but enhanced at the maximum of the spectrum.
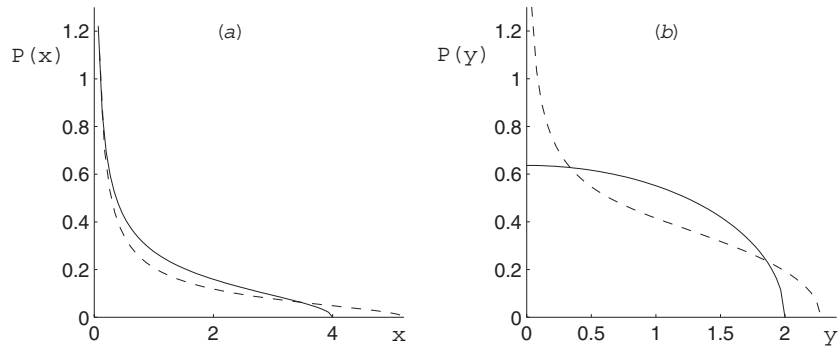
Figure 1. Level density of random density matrices $P(x)$ for Bures measure (dashed line) and Hilbert–Schmidt measure (solid line) (a). The latter becomes a quarter-circle distribution in the rescaled variable, $y = \sqrt{x}$ (b).

The above distributions may be derived in an alternative way by minimization of the action functional

$$A_{\mathrm{HS}} = - \int \mathrm{d}x \, \mathrm{d}x' \, P(x) P(x') \ln|x - x'| \tag{4.5}$$

for the HS measure, and

$$A_{\mathrm{B}} = A_{\mathrm{HS}} + \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}x' \, P(x) P(x') \ln(x + x') \tag{4.6}$$

for the Bures measure. Both (unknown) solutions of these minimization problems should satisfy the normalization condition, $\int P(x) \, \mathrm{d}x = 1$ and the relation induced by the unit trace constraint, $\int x P(x) \, \mathrm{d}x = 1$. Both conditions can be implemented with the help of Lagrange multipliers. The resulting integral equations for $P(x)$ may be solved by the Green function

$$G(t) = \int \mathrm{d}x \frac{P(x)}{x - t} \tag{4.7}$$

where the cut along the real axis gives the densities (4.3), (4.4). The Hilbert–Schmidt measure leads to a rather simple Green function

$$G_{\mathrm{HS}}(t) = \frac{1}{2} \left(\sqrt{1 - \frac{4}{t}} - 1 \right), \tag{4.8}$$

for $t < 0$ and otherwise given by analytic continuation, corresponding to (4.3). The Green function corresponding to the Bures measure is more complicated

$$G_{\mathrm{B}}(t) = \frac{1}{6} \left(z + \frac{1}{z} - 1 \right) \qquad \text{with} \qquad z = \left(-\frac{a}{t} \right)^{2/3} \left(1 - \sqrt{1 - \left(\frac{t}{a} \right)^2} \right)^{2/3} \tag{4.9}$$

for $-a < t < 0$ (otherwise given by analytic continuation) and leads to the distribution (4.4). The Green functions fulfil the generalized *Pastur equations* [16]

$$G_{\mathrm{HS}}(t) = \frac{-1/t}{1 + G_{\mathrm{HS}}(t)} \qquad \text{and} \qquad G_{\mathrm{B}}(t) = \frac{-1/t}{\sqrt{1 + 2G_{\mathrm{B}}(t)}}, \tag{4.10}$$

which suggests an interpolation formula between the Hilbert–Schmidt measure ($\alpha = 1$) and the Bures measure ($\alpha = 2$):

$$G_{\alpha}(t) = \frac{-1/t}{[1 + \alpha G_{\alpha}(t)]^{1/\alpha}}. \tag{4.11}$$

It would be interesting to analyse the family of interpolating measures which lead to the above Green functions.

5. Eigenvalue density and eigenvalue correlation for induced measures

In this section we are going to investigate statistical properties of the induced measures (2.3) defined in the space $\mathcal{M}_N$ of density matrices of size N. An integer $K \geqslant N$, represents the size of an environment and may be treated as a parameter labelling the measure.

5.1. Eigenvalue density

The one-point density $P(\lambda)$ is obtained from the Green function

$$G(\lambda) = \left\langle \frac{1}{N} \sum_i \frac{1}{\lambda_i - \lambda} \right\rangle_{N,K} \qquad \text{with} \qquad P(\lambda) = \frac{1}{\pi} \operatorname{Im} G(\lambda + i\delta), \quad (5.1)$$

and the Green function will be derived from the generating function

$$Z(\mu) = \left\langle \prod_i \left(\frac{\lambda_i - \mu}{\lambda_i - \lambda} \right) \right\rangle_{N,K} \qquad \text{with} \qquad G(\lambda) = -\frac{1}{N} \frac{\partial}{\partial \mu} Z|_{\mu=\lambda}. \quad (5.2)$$

Due to the structure of $P_{N,K}$ with the van der Monde determinant we may write $Z(\mu)$ as inverse Laplace transform of a determinantal function:

$$Z(\mu) \propto \int_{-i\infty+\epsilon}^{+i\infty+\epsilon} \frac{\mathrm{d}s}{2\pi \mathrm{i}} \, \mathrm{e}^s \det \left(\int_0^\infty \mathrm{d}x \, \mathrm{e}^{-sx} x^{K-N+i+j-2} \left(\frac{x-\mu}{x-\lambda} \right) \right) \quad (5.3)$$

with $i, j = 1, 2, \ldots, N$. From this we immediately obtain

$$G(\lambda) = \frac{\Gamma(KN)}{N} \int_{-i\infty+\epsilon}^{+i\infty+\epsilon} \frac{\mathrm{d}s}{2\pi \mathrm{i}} \frac{\mathrm{e}^s}{s^{KN}} \sum_{i,j=1}^N W_{j,i}^{-1} \int_0^\infty \mathrm{d}x \frac{\mathrm{e}^{-sx} x^{K-N+i+j-2}}{(x-\lambda) s^{K-N+i+j-1}}. \quad (5.4)$$

Here the matrix W of size N is given by

$$W_{i,j} = \Gamma(K - N + i + j - 1). \quad (5.5)$$

An explicit form of the matrix W^{-1} reads

$$(W^{-1})_{i,j} = (-1)^{i+j} \sum_{k=\max(i,j)}^N \binom{k-1}{i-1} \binom{k-1}{j-1} \frac{\Gamma(K-N+k)}{\Gamma(k)\Gamma(K-N+i)\Gamma(K-N+j)}. \quad (5.6)$$

It turns out that the x integral in equation (5.4) contributes only for $x < 1$ and we obtain the result

$$G(\lambda) = \frac{\Gamma(KN)}{N} \sum_{i,j=1}^N W_{j,i}^{-1} \int_0^1 \mathrm{d}x \frac{x^{K-N+i+j-2}(1-x)^{KN-(K-N+i+j)}}{\Gamma[KN - (K-N+i+j-1)](x-\lambda)}. \quad (5.7)$$

The normalization constant in equations (5.4), (5.7) has been restored by the known asymptotic behaviour of $G(\lambda)$. Thus we immediately obtain the density

$$P(\lambda) = \frac{\Gamma(KN)}{N} \sum_{i,j=1}^N W_{j,i}^{-1} \frac{\lambda^{K-N+i+j-2}(1-\lambda)^{KN-(K-N+i+j)}}{\Gamma[KN - (K-N+i+j-1)]}. \quad (5.8)$$

The above rather complicated form of obtaining the density already derived by Page [10], allows us to calculate the moments with the help of Euler's beta-function

$$\langle \lambda^q \rangle = \frac{\Gamma(KN)}{N\Gamma(KN+q)} \sum_{i,j=1}^{N} W_{j,i}^{-1} \Gamma(K-N+i+j-1+q). \tag{5.9}$$

From this follows the mean von Neumann entropy

$$\langle S \rangle = -N\langle \lambda \ln \lambda \rangle = \psi(KN+1) - \frac{1}{KN} \sum_{i,j} W_{j,i}^{-1} \Gamma(K-N+i+j)\psi(K-N+i+j) \tag{5.10}$$

where $\psi(x)$ is Euler's digamma-function $= \Gamma'(x)/\Gamma(x)$, and where we have suppressed the index $_{N,K}$ at the angular brackets. The result for the average entropy has been conjectured by Page [10], and later proved in [17–19]. It is a rational number. All the formulae are valid for $K \geqslant N$. Again for $K < N$ one has to interchange K and N, obtaining density and moments of the K positive eigenvalues.

One may explicitly give some average traces over the induced measure $\mu_{N,K}$:

$$\langle \mathrm{Tr}\, \rho^2 \rangle = \frac{K+N}{KN+1}, \qquad \langle \mathrm{Tr}\, \rho^3 \rangle = \frac{(K+N)^2 + KN + 1}{(KN+1)(KN+2)}, \tag{5.11}$$

$$\langle \mathrm{Tr}\, \rho^4 \rangle = \frac{(K+N)[(K+N)^2 + 3KN + 5]}{(KN+1)(KN+2)(KN+3)}, \tag{5.12}$$

the first of which has appeared already in the paper of Lubkin [8], while the others are consistent with the recent work of Malacarne *et al* [20]. To find the coefficients of the polynomial in the denominator it is useful to know its order and symmetry as can be found going back to a Gaussian integral writing the density matrix as a matrix of Wishart form $\rho = \psi\psi^\dagger$. In principle they are contained in formula (5.9).

5.2. Eigenvalue correlation

The eigenvalue correlation can be obtained from the Green function correlation

$$\left\langle \frac{1}{N} \sum_i \frac{1}{\lambda_i - \lambda} \frac{1}{N} \sum_i \frac{1}{\lambda_i - \mu} \right\rangle, \tag{5.13}$$

which can be derived from the generating function

$$\left\langle \prod_i \left(\frac{(\lambda_i - \kappa_1)(\lambda_i - \kappa_2)}{(\lambda_i - \lambda)(\lambda_i - \mu)} \right) \right\rangle. \tag{5.14}$$

The result for the two-eigenvalue density $P(\lambda, \mu)$, which can be obtained along the same lines as in section 5.1 is then

$$P(\lambda, \mu) = \theta(1 - \lambda - \mu) \frac{\Gamma(KN)}{N(N-1)} \sum_{i,j,k,l=1}^{N} \left[W_{j,i}^{-1} W_{l,k}^{-1} - W_{l,i}^{-1} W_{j,k}^{-1} \right]$$

$$\times \frac{\lambda^{K-N+i+j-2} \mu^{K-N+k+l-2} (1 - \lambda - \mu)^{KN-2K+2N-i-j-k-l+1}}{\Gamma(KN - 2K + 2N - i - j - k - l + 2)}. \tag{5.15}$$

We have checked $P(\lambda) = \int P(\lambda, \mu)\, d\mu$. In order to prove this, scale μ with $1 - \lambda$, integrate over μ with the help of Euler's beta-function and use equation (5.5). Then the factor $N-1$ in the denominator of equation (5.15) cancels and the result is equation (5.8). The first bracket

under the sum in (5.15) ensures level repulsion $\propto (\lambda - \mu)^2$ for $\lambda - \mu \to 0$. Furthermore there is additional repulsion from the boundaries at $\lambda = 0$, $\mu = 0$, $1 - \lambda - \mu = 0$.

It is again easy to calculate the moments with the help of Euler's beta-function

$$\langle \lambda^L \mu^M \rangle = \frac{\Gamma(KN)}{N(N-1)\Gamma(KN+L+M)} \sum_{i,j,k,l=1}^{N} \left[W_{j,i}^{-1} W_{l,k}^{-1} - W_{l,i}^{-1} W_{j,k}^{-1} \right]$$
$$\times \Gamma(K-N+i+j-1+L)\Gamma(K-N+k+l-1+M). \tag{5.16}$$

For the entropy correlation we have

$$\langle SS \rangle = N(N-1)\langle \lambda(\ln\lambda)\mu(\ln\mu)\rangle + N\langle \lambda^2(\ln\lambda)^2\rangle, \tag{5.17}$$

which can be obtained by double differentiation of $\langle \lambda^L \mu^M \rangle$ and $\langle \lambda^{L+M} \rangle$ with respect to L and M at $L = M = 1$. Again one may obtain formulae for $K < N$ by interchange of K and N.

6. Concluding remarks

It is well known that that there is no single, naturally distinguished probability measure in the set of mixed quantum states of a fixed size N. Guessing a mixed state randomly without any additional information whatsoever, it is legitimate to use the Bures measure (2.4), related to the statistical distance and distinguishability. On the other hand, if it is known that the mixed state has arisen by the partial tracing over a K-dimensional environment, one uses the induced measure (2.3), which reduces to the Hilbert–Schmidt measure in the special case $K = N$.

In this work we investigated statistical properties of ensembles of density matrices of a fixed size generated according to the Bures or the Hilbert–Schmidt measure. We computed the averages over the set of mixed quantum states with respect to both measures and derived the level density in the asymptotic limit of large matrices. Furthermore, for measures obtained from random pure states of a composite system by partial tracing we computed the one-point eigenvalue density, the exact two-point eigenvalue density, the corresponding moments and average entropies. On one hand, results concerning average traces and average entropy may be useful from the point of view of the theory of quantum information [21]: the von Neumann entropy of a mixed state ρ is equal to the entanglement of the pure state $|\Psi\rangle$ belonging to a composed Hilbert space, which purifies ρ. On the other hand, results obtained contribute to the theory of random matrices: the ensembles of random density matrices distributed according to the Bures measure display different properties then the standard Gaussian ensembles of Wigner and Dyson [11].

It is worth adding that the ensembles of random states analysed in this work do not cover all the cases of a physical interest. For instance, it is natural to assume that in a concrete experiment a mixed state ρ is formed by applying a known quantum channel Φ (completely positive, trace preserving map) on a random pure state,

$$\rho = \Phi(|\psi\rangle\langle\psi|). \tag{6.1}$$

Without any information concerning the pure state $|\psi\rangle \in \mathcal{H}_N$ one has to assume that it is generated according to the natural Fubini–Study measure in the set of pure states. In this manner any quantum channel Φ induces by (6.1) a certain measure in the space of mixed quantum states. Hence it would be interesting to repeat the computation performed in this work for ensembles of mixed states obtained by physically motivated quantum channels. Such research will be a subject of a forthcoming publication.

Acknowledgments

Appendix. Moments for Bures measure

One may derive all moments for the Bures distribution $P_B(\rho)$ from a Laguerre-type ensemble

$$P_B^L(\rho) \propto \theta(\rho) \, e^{-\text{Tr}\,\rho} \prod_{i,j}^{1,\ldots,N} (\lambda_i + \lambda_j)^{-1/2}, \tag{A.1}$$

where λ_i denote eigenvalues of ρ. Moments are related by

$$\int M_p(\rho) P_B(\rho) D\rho = \frac{\Gamma(N^2/2)}{\Gamma(N^2/2 + p)} \int M_p(\rho) P_B^L(\rho) D\rho, \tag{A.2}$$

where $M_p(\rho)$ is a homogeneous function of ρ of degree p and $D\rho$ is the matrix volume element of a Hermitian matrix. Next we may write

$$P_B^L(\rho) \propto \theta(\rho) \int DA \, e^{-\text{Tr}[\rho(1+A^2)]} \tag{A.3}$$

with a Hermitian matrix A. Let us denote its eigenvalues by $\{A_i\}$. In the following we use the formula [13]:

$$\theta(\rho) \, e^{-\text{Tr}(\rho\epsilon)} = B_N \det(\delta/\delta\rho + \epsilon)^{-N} \delta(\rho) \tag{A.4}$$

with a positive definite Hermitian matrix ϵ and

$$B_N = \pi^{N(N-1)/2} \Gamma(1)\Gamma(2) \cdots \Gamma(N). \tag{A.5}$$

Note that on the left-hand side of equation (A.4) there is the restriction $\rho \geqslant 0$, while on the right-hand side we have no restriction on integration for ρ. With the above formula it is easy to compute the matrix Laplace transform

$$\int e^{-\text{Tr}(E\rho)} P_B^L(\rho) D\rho \propto \int DA \det(E + 1 + A^2)^{-N} \tag{A.6}$$

with a non-negative matrix E of size N. For $E = 0$ this formula leads to the normalization constant for the Bures measure [13]. Let $\{E_1, \ldots, E_N\}$ denote the eigenvalues of E. With the help of the Itzykson–Zuber integral [22, 23] the right-hand side of equation (A.6) is proportional to

$$\int dA_1, \ldots, dA_N \, \frac{[\Delta(A)]^2}{\Delta(A^2)\Delta(-E)} \det\left[\frac{1}{1 + A_i^2 + E_j}\right] \tag{A.7}$$

with the van der Monde determinant $\Delta(A) = \prod_{i<j}(A_i - A_j)$. Finally one may perform the A_i integrations in the complex plane, arriving at the generating function

$$Z_B^L(E) = \int e^{-\text{Tr}(E\rho)} P_B^L(\rho) D\rho = \prod_{i,j}^{1,\ldots,N} \frac{2}{\sqrt{1 + E_i} + \sqrt{1 + E_j}}. \tag{A.8}$$

This expression has a rather simple expansion in powers of E. It starts like

$$Z_B^L(E) = 1 - \frac{N}{2} \sum E_i + \left(\frac{N^2}{8} + \frac{1}{16}\right)\left(\sum E_i\right)^2 + \frac{3N}{16} \sum E_i^2 + O(E^3). \tag{A.9}$$

Thus it is possible to obtain all moments by matrix derivation, e.g.

$$\langle \text{Tr}\, F(\rho)\rangle_{\text{B}}^{\text{L}} = \text{Tr}\, F(-\delta/\delta E)Z_{\text{B}}^{\text{L}}(E)|_{E=0}. \tag{A.10}$$

The corresponding generating function for the Hilbert–Schmidt measure is even more simple and reads

$$Z_{\text{HS}}^{\text{L}}(E) = \int e^{-\text{Tr}\,(E\rho)} P_{\text{HS}}^{\text{L}}(\rho)\, D\rho = \prod_{i=1}^{N} \frac{1}{(1+E_i)^N}. \tag{A.11}$$

Its expansion starts like

$$Z_{\text{HS}}^{\text{L}}(E) = 1 - N \sum E_i + \frac{N^2}{2}\left(\sum E_i\right)^2 + \frac{N}{2}\sum E_i^2 + O(E^3). \tag{A.12}$$

To obtain the matrix derivatives $\delta/\delta E$ is not so easy in general, since everything is expressed in eigenvalues E_i e.g.

$$\langle \text{Tr}\, \rho^q\rangle_{\text{B}}^{\text{L}} = \text{Tr}\,(-\delta/\delta E)^q Z_{\text{B}}^{\text{L}}(E)|_{E=0} = \frac{1}{\Delta(E)}\sum_{i=1}^{N}(-\partial/\partial E_i)^q[\Delta(E)Z_{\text{B}}^{\text{L}}(E)]|_{E=0}. \tag{A.13}$$

One can proof the last equation with the help of the Itzykson–Zuber integral. The above formulae were used to derive the few results for Bures moments in section 3.

References

[1] Życzkowski K and Sommers H-J 2001 *J. Phys. A: Math. Gen.* **34** 7111
[2] Hall M J W 1998 *Phys. Lett.* **A 242** 123
[3] Bures D J C 1969 *Trans. Am. Math. Soc.* **135** 199
[4] Uhlmann A 1992 *Groups and Related Topics* ed R Gielerak *et al* (Dodrecht: Kluwer)
[5] Petz D and Sudár C 1996 *J. Math. Phys.* **37** 2662
[6] Braunstein S L and Caves C M 1994 *Phys. Rev. Lett.* **72** 3439
[7] Uhlmann A 1995 *Rep. Math. Phys.* **36** 461
[8] Lubkin E 1978 *J. Math. Phys.* **19** 1028
[9] Lloyd S and Pagels H 1988 *Ann. Phys., NY* **188** 186
[10] Page D 1993 *Phys. Rev. Lett.* **71** 1291
[11] Mehta M L 1991 *Random Matrices* 2nd edn (New York: Academic)
[12] Slater P B 1999 *J. Phys. A: Math. Gen.* **32** 8231
[13] Sommers H-J and Życzkowski K 2003 *J. Phys. A: Math. Gen.* **36** 10083
[14] Życzkowski K and Sommers H-J 2003 *J. Phys. A: Math. Gen.* **36** 10115
[15] Sengupta A M and Mitra P P 1999 *Phys. Rev.* E **60** 3389
[16] Pastur L A 1972 *Theor. Math. Phys.* **10** 67
[17] Foong S K and Kanno S 1994 *Phys. Rev. Lett.* **72** 1148
[18] Sánchez-Ruiz J 1995 *Phys. Rev.* E **52** 5653
[19] Sen S 1996 *Phys. Rev. Lett.* **77** 1
[20] Malacarne L C, Mandes R S and Lenzi E K 2002 *Phys. Rev.* E **65** 046131
[21] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[22] Itzykson C and Zuber J B 1980 *J. Math. Phys.* **21** 411
[23] Chandra H 1957 *Am. J. Math.* **79** 87